MODULARITY LIFTING THEOREMS OBERSEMINAR PROGRAM

1. Introduction

The aim of this study group is to understand the ideas appearing in the modularity lifting techniques of Taylor–Wiles (with improvements by Diamond, Fujiwara, Kisin, etc..). For this we'll follow the Arizona Winter School notes of Toby Gee [Gee], which work in the setting of two dimensional representations over a totally real field F.

These results are part of the Langlands program, which predicts a relationship between (certain) modular or automorphic forms over F and (certain) representations of $G_F = \mathrm{Gal}(\overline{F}/F)$. The "automorphic to Galois" direction of this correspondence involves attaching Galois representations to automorphic forms and, in our setting, this has been known since the 90's. The other direction is more difficult; one has to show every (expected) Galois representation arises from this construction. The standard strategy is to first show that a given Galois representation is congruent modulo p to an automorphic Galois representation, and then show such a congruence implies the original representation was automorphic. The first step is a version of Serre's modularity conjecture and is very hard; currently it is only known for 2-dimensional representations of $G_{\mathbb{Q}}$. Results which give the second step are the modularity lifting theorems we will be interested in.

The precise statement we will try to understand is:

Theorem 1.1. *Recall F is a totally real field. Let $\mathcal{O}$ be the ring of integers in a sufficiently large finite extension of $\mathbb{Q}_p$ with $p > 3$. Let $\rho, \rho_0 : G_F \to \mathrm{GL}_2(\mathcal{O})$ be continuous homomorphisms with $\rho \equiv \rho_0$ modulo $\mathfrak{m}_{\mathcal{O}}$. Assume that ρ_0 is modular and that ρ is geometric (i.e. satisfies prescribed local conditions). Assume further that*

(1) *ρ and ρ_0 have the same Hodge–Tate weights, and over each embedding $\sigma : F \to \mathcal{O}[\frac{1}{p}]$ these consist of pairs of distinct integers.*
(2)
- *p is unramified in F*
- *$\rho|_{G_{F_v}}$ and $\rho_0|_{G_{F_v}}$ are crystalline for every prime $v \mid p$ of F.*
- *The Hodge–Tate weights of ρ and ρ_0 differ by at most $p-2$.*

(3) $\mathrm{Im}(\overline{\rho}) \supset \mathrm{SL}_2(\mathbb{F}_p)$ *(in particular $\overline{\rho}$ is absolutely irreducible).*

Then ρ is modular.

See [Gee, Section 6] for a discussion on the extent with which these conditions can be relaxed. To prove such a theorem one establishes an "$R = T$" statement. Here R is a deformation ring classifying liftings of $\overline{\rho}$ with the prescribed local conditions, while T is a Hecke algebra acting on a suitable space of modular forms. The construction of Galois representations attached to these modular forms furnishes a map $R \to T$ and the goal is to show this is an isomorphism. This is achieved by a process called patching which reduces the problem to one involving the geometry of (more accessible) local deformation rings.

2. Program

Talk 1 : Introduction and overview (05 April, Robin).

Talk 2: Grothendieck's l-adic monodromy theorem (12 April, Judith). The goal of this talk is to cover [Gee, Section 2.7]. In particular, you should prove [Gee, Proposition 2.18] following [Gee, Exercise 2.20] (for a complete proof see also [FO, Theorem 1.24]).

Talk 3: Galois deformation rings (19 April, Stefania). Cover the general statements regarding framed and unframed deformation rings from [Gee, Section 3.1], and give some examples (e.g. [Sho16, Lemma 2.5]). You don't need to give a proof of the representability.

The second task is to cover the cohomological descriptions of the tangent spaces and lifting obstructions from [Gee, Section 3.10].

Talk 4: Deformation conditions and some local deformation rings for $l \neq p$ (26 April, Martina). There are two parts for this talk. The first is to continue the discussion on general deformation rings by explaining the formalism of a deformation condition from [Gee, Section 3.15]. Also discuss the example of a deformation condition given by fixing the determinant [Gee, Section 3.18].

The second part is to describe the local deformation rings appearing later on for $l \neq p$. This is contained in [Gee, Section 3.29] up to the end of Section 3, but we don't need everything here. The key things you should cover are [Gee, Lemma 3.33] and the proof explained in [Gee, Exercise 3.34] (which is essentially the proof of [Sho16, 5.3] but without fixing determinants), and statement of parts (1) and (3) concerning $R^{\square}_{\bar{\rho},\chi,1}$ and $R^{\square}_{\bar{\rho},\chi,\zeta}$ from [Gee, Theorem 3.38].

Talk 5: Local deformation rings when $l = p$ (03 May, Luisa). Following [Gee, 2.21], give an overview of the hierarchy of local p-adic Galois representations using p-adic Hodge theory. The rest of the talk should then discuss Theorem 3.28 and sketch some of its proof, following [CHT08, Section 2.4.1]. Here you can use the Fontaine–Laffaille theory described in the first part of [CHT08, Section 2.4.1] as a black box. Then formal smoothness is essentially [CHT08, Lemma 2.4.1] and the dimension calculation is [CHT08, Corollary 2.4.3].

Talk 6: Presenting global deformation rings over local ones (10 May, Konrad). Give the formalism of global deformation problems with local conditions as in [Gee, 3.20]. Then explain as much of the proof of [Gee, Proposition 3.24] as possible. Additional references which could be helpful are [Böc, Lecture 5] and [Kis, Lecture 4].

Talk 7: Modular forms on quarternion algebras (17 May, Claudius). Give an introduction to quarternion algebras over a number field F, including their classification up to isomorphism in terms of their split places. See for example [Hid06, Section 2.1.1]. Define the spaces of modular forms on D as in [Gee, 4.8]. Explain how this notion recovers the classical notion of a modular form when $F = \mathbb{Q}$ and $D = M_2(\mathbb{Q})$ as in [Gee, Exercise 4.9]. In the remaining time describe some spaces of modular forms for definite D (e.g. weight 2 with trivial character as in the last paragraph of [Gee, p.25]).

Talk 8: Automorphic representations (24 May, Peter). Recall the classification of smooth irreducible representations of $\mathrm{GL}_2(K)$ for a local field K as in [Gee, Section 4.1] (but you don't need to say anything about the local Langlands correspondence). Also discuss [Gee, Section 4.6, Exercise 4.7].

In the second part of the talk discuss the notion of an admissible automorphic representation in [Gee, Definition 4.10] and [Gee, Fact 4.16]. Also, say what you can about the description of irreducible such representations as restricted tensor products ([Gee, Fact 4.11]). A proof is given in [Fla79], with more details in [Bum97, Section 3.4].

Talk 9: Multiplicity one and new forms (31 May, Kieu Hieu). In this talk you should state [Gee, Facts 4.15 and 4.16]. Their statements are all that is needed for the rest of the study group, so the goal for the rest of the talk is to give background and some of the ideas going into their proofs. Here are the references I know:

For the theory of newforms: For the classical notion of a newform see e.g. [DS05, Section 5.6]. The automorphic point of view follows from a purely local result [Cas73, Theorem 1]. See also [Del73, Section 2.2] for a reasonably quick proof granting the existence of Kirillov models.

For (strong) multiplicity one: a proof is sketched in [PS79] (for GL_n). A proof is also given (in the classical setting of $F = \mathbb{Q}$ and split D) in [Del73, Theorem 2.5.6], with more details.

Talk 9.5: Break for 07 - 10 June holiday.

Talk 10: Galois representations attached to modular forms, and the statement of modularity lifting (14 June, Eva). The first goal of this talk is to state how to attach Galois representations to modular forms for GL_2 as in [Gee, Fact 4.20]. For this we will also need the description of the reciprocity map from the local Langlands correspondence (as described in the second part of [Gee, Section 4.1]). You can then give the statement of the main result of the study group [Gee, Theorem 5.2]. You should then explain the reduction steps made in [Gee, Section 5.5] using cyclic base change ([Gee, Fact 4.22]) as a black box.

Talk 11: Jacquet–Langlands and some p-adic automorphic forms on definite quaternion algebras (21 June, Alex). The goal of this talk is to explain how the modularity in the main theorem can be viewed in terms of modular forms on a definite quaternion algebra. For this state the Jacquet–Langlands correspondence ([Gee, Fact 4.17]). If you have time you could discuss some of the examples from [JLnotes]. Then explain the construction of p-adic modular forms on such definite quaternion algebras from [Gee, Section 5.3] finishing with [Gee, Proposition 5.4].

Talk 12: Patching (28 June, Yifei). Cover [Gee, Section 5.6], concluding with [Gee, Proposition 5.9].

Talk 13: Patching continued (05 July, Emanuele). Sketch the proof of the existence of the Taylor–Wiles primes in [Gee, Proposition 5.10] and then finish [Gee, Section 5] and the proof of the main theorem.

References

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[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005.

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[Sho16] Jack Shotton, *Local deformation rings for* GL_2 *and a Breuil-Mézard conjecture when* $\ell \neq p$, Algebra Number Theory **10** (2016), no. 7, 1437–1475.